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μ-Holomorphy conditions in 2D conformal models

M. Kachkachi*

Réseau National de Physique Théorique, Rabat, Morocco

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Abstract

Here, we give an explicit study of the μ-holomorphic j-differentials in conformal geometry and as application in 2D conformal models. Moreover, we rewrite the anomalous conformal Ward identity as certain kind of a deformed μ-holomorphy condition. Indeed, the conformal Ward identity is expressed as μ-holomorphy equation.

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1. Introduction

Two-dimensional conformal field theories on Riemann surfaces without boundary are the relevant models in string theory [1]. Recently, their dependence on the background geometry has been exploited to obtain effective actions for two-dimensional quantum gravity. This has led to exciting developments in non-critical string theory [2] and may conceivably shed some light on the quantization program of highest dimensional gravity.

Most of the study on the subject are concerned with Lagrangian field theories on a two-dimensional manifold (Σ, g) which are both Weyl and diffeomorphisms invariant at the classical level [3,4]. The quantization program is carried out by means of diffeomorphism invariant scheme. In general, however, a Weyl anomaly emerges in this way whose strength is measured by a real coefficient k; the central charge of the model under consideration, up to a conventional normalization. The form of such anomaly is universal. The Weyl anomaly can be eliminated by either (i) constraining the field content of the model so that the central

* Present address: Université Hassan 1er, Faculté des Sciences and Techniques, Département Mathématiques and Informatique, UPM, Settat, Morocco. Tel.: +212-3-40-0736; fax: +212-3-40-0969.
E-mail address: kachkac@ibnsina.uh1.ac.ma (M. Kachkachi).

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charge vanishes as in the case of string theory [3], or (ii) substracting from the effective action a suitable local counterterm that absorbs the Weyl anomaly at the cost of creating a diffeomorphism anomaly [5] whose strength is again measured by the central charge k. The form of such counterterm is also universal. The silent feature of the diffeomorphism anomaly is that it is chirally split, it is the sum of two terms each of which is the complex conjugate of the other. This fact is intimately related to the holomorphic factorization property of the Weyl invariant effective action [6,7] that can be written as

$$\Gamma_{\mathrm{W}}(\mu, \bar{\mu}; R_0, \overline{R_0}) = \Gamma_{\mathrm{p}}(\mu; R_0) + \overline{\Gamma_{\mathrm{p}}(\mu; R_0)}, \tag{1.1}$$

where R_0 is the holomorphic projective connection, i.e. $\bar{\partial} R_0 = 0$ in a reference holomorphic coordinates system $(z, \bar{z})$. Γ_{p} is called the Polyakov action of the model under consideration [8]. In this Beltrami parameterization scheme, the metric g can be parameterized as follows:

$$g = \exp(\varphi)\rho_0 |\mathrm{d}z + \mu\, \mathrm{d}\bar{z}|^2, \tag{1.2}$$

where φ is the Weyl phase, μ the Beltrami differential characterizing the conformal class of the metric g, ρ_0 the background metric and is needed to write g invariantly.

The action Γ_{p} is a most fundamental object. It depends holomorphically on the Beltrami differential and on the background R_0. It satisfies the chiral conformal Ward identity:

$$s\Gamma_{\mathrm{p}}(\mu; R_0) = kA(C; \mu; R_0), \tag{1.3}$$

where s is the nilpotent BRST symmetry's generator and $C = c + \mu\bar{c}$ the combination of the associated diffeomorphisms parameters ghosts. In this BRST formalism, the holomorphic projective connection R_0 is s-invariant, i.e. $sR_0 = 0$.

In the diffeomorphism Lie algebra formalism, the conformal Ward identity (1.3) expresses the anomalous break down of the diffeomorphism symmetry [9]:

$$W_2 \frac{\delta \Gamma_{\mathrm{p}}}{\delta \mu} = \frac{-k}{12\pi} L_3^{R_0}(\mu), \tag{1.4}$$

where $W_2 \equiv \bar{\partial} - \mu\partial - 2\partial\mu$ is the Ward operator and $L_3^{R_0} \equiv \partial^3 + 2R_0\partial + \partial R_0$ the third Bol's operator [10,11] associated to the holomorphic projective connection R_0.

The independence of $\Gamma_{\mathrm{p}}(\mu; R_0)$ from φ and ρ_0 entails that such function depends only on the background conformal geometry parameterized by the Beltrami differential μ. This suggests that a natural scheme for the study of two-dimensional conformal field theories on Riemann surfaces should rely ab initio and exclusively on conformal geometry [12,13]. Moreover, combinations of Polyakov actions can be used to construct explicitly chiral conformal theories. Then, any Polyakov action may serve as a "classical" action for 2D quantum gravity in the light cone gauge [8].

2. Complex and projective structures on Σ

Let us consider a Riemann surface Σ equipped with an atlas of compatible complex analytic coordinates system, i.e. we have complex coordinates z_α defined on each patch and the transition functions $f_{\alpha\beta}$ between two patches are holomorphic:

$$z_\alpha = f_{\alpha\beta}(z_\beta). \tag{2.1}$$

We may define another atlas of compatible complex analytic coordinates z'_α. The transition functions $z'_\alpha = f'_{\alpha\beta}(z'_\alpha)$ are again holomorphic. Moreover, if z'_α and z_β are related by holomorphic (local) diffeomorphism they are said to belong to the same complex structure. However, if the transition functions $z'_\alpha = f_{\alpha\beta}(z_\beta)$ are not holomorphic, the two complex structures (z_α) and (Z'_α) belong to different atlas. Now, a Riemann surface Σ is a C^∞-differentiable bi-dimensional manifold endowed with a reference complex structure which is fixed by an analytic complex structure $(z_\alpha, \overline{z_\alpha})$. On this surface, we consider a Beltrami differential $\mu(z, \bar{z})$ that induces another complex structure on Σ parameterized by local coordinates $(Z_\alpha, \overline{Z_\alpha})$. These latters are C^∞-diffeomorphisms of the reference variables (z_α) and satisfy, in each map (U_α, z_α), the Beltrami equation:

$$(\partial_{\overline{z_\alpha}} - \mu_{\overline{z_\alpha}}^{z_\alpha}(z_\alpha)\partial_{z_\alpha})Z_\alpha(z_\alpha), \tag{2.2}$$

where $\mu \in C^\infty(\Sigma)$ and $|\mu| \prec 1$.

For simplicity, we omit the indice α characterizing the map and we write $\partial \equiv \partial_{z_\alpha}$ and c.c. One can verify that Eq. (2.2) enables us to express the one form of the coordinate Z as follows:

$$\mathrm{d}Z = \lambda_z^Z(\mathrm{d}z + \mu\,\mathrm{d}\bar{z}), \tag{2.3}$$

with $\lambda \equiv \lambda_z^Z = \partial Z$ is an integrating factor non-local in μ. The diffeomorphism

$$z \to Z(z, \bar{z}), \qquad \mu = 0 \to \mu \neq 0, \quad (\bar{\partial} - \mu\partial)Z = 0 \tag{2.4}$$

is called a quasi-conformal transformation that becomes conformal for $\mu = 0$. It can be seen as the transition from the reference conformal structure ($\mu = 0$) to the other one determined by $\mu = \bar{\partial}Z/\partial Z$. The particular case of holomorphic transition functions between two compatible complex analytic coordinates (z) and (ω) is the projective transition function which is defined by

$$z \to \omega(z), \qquad \omega \in SL(2, C) \tag{2.5}$$

where $SL(2, C)$ is the Mobius group. The atlas of compatible complex analytic structures whose transition functions are projective is called a projective atlas.

3. Diffeomorphisms, quasi-conformal, conformal and projective transition functions on Σ

To characterize such transformations, let us consider the Schwarzian derivative $\zeta_z(\omega)$ of the function ω with respect to z which is defined by [9]

$$\zeta_z(\omega) \equiv \partial^2 \ln \partial\omega - \tfrac{1}{2}(\partial \ln \partial\omega)^2. \tag{3.1}$$

Then, we have the following properties:

(I) If the transition function $z \to \omega(z, \bar{z})$ is any diffeomorphism, then the Schwarzian derivative $\zeta_z(\omega)$ satisfies the equation

$$(\partial^2 + \tfrac{1}{2}\zeta_z(\omega))((\partial\omega)^{-1/2}) = 0. \tag{3.2}$$

(J) If $z \to \omega(z, \bar{z})$ is quasi-conformal; $\bar{\partial}\omega = \mu\partial\omega$ then we have

$$\bar{\partial}\zeta_z(\omega) = L_3^{\zeta_z(\omega)}(\mu). \tag{3.3}$$

One can see that Eq. (3.3) is nothing but the μ-holomorphy equation for $\zeta_z(\omega)$ [11].

(K) If the diffeomorphism $z \to \omega(z, \bar{z})$ is holomorphic ($\bar{\partial}\omega = 0$), the Schwarzian derivative is also holomorphic:

$$\bar{\partial}\zeta_z(\omega) = 0. \tag{3.4}$$

(L) If $z \to \omega(z)$ is projective, $\zeta_z(\omega)$ vanishes:

$$\zeta_z(\omega) = 0. \tag{3.5}$$

Now, by combining Eqs. (1.2) and (2.3) we express the metric g in terms of isothermal coordinates (Z):

$$g = \exp(\varphi)|\mathrm{d}Z|^2 \tag{3.6}$$

which are the solutions of the Beltrami equation

$$(\bar{\partial} - \mu\partial)Z = 0. \tag{3.7}$$

Moreover, this equation determines Z up to a holomorphic reparameterization $Z \to F(Z)$. Indeed, one can verify the following equation:

$$(\bar{\partial} - \mu\partial)F = \bar{\lambda}(1 - \mu\bar{\mu})\partial_{\bar{Z}}F = 0. \tag{3.8}$$

Hence, to any given μ one can associate exactly one complex structure which is provided by solutions of (3.8). Conversly, for given F one can define $\mu \equiv \bar{\partial}F/\partial F$ and if G is any holomorphic function of F one has

$$\mu = \frac{\bar{\partial}G}{\partial G} = \frac{\bar{\partial}F}{\partial F}. \tag{3.9}$$

On the other hand, one can verify that the Schwarzian derivative (3.1) is invariant under Mobius subgroup of holomorphic diffeomorphisms. However, under a holomorphic diffeomorphism $z \to z' = f(z)$, $\zeta_z(\omega)$ transforms as

$$\zeta_z(\omega) \to (\partial f)^{-2}(\zeta_z(\omega) - \zeta_z(z')). \tag{3.10}$$

Furthermore, as μ is a true $(-1, 1)$ differential with respect to a conformal change of coordinates:

$$\mu_z \to \mu_{z'} = (\partial z')(\overline{\partial z'})\mu_z, \tag{3.11}$$

it is easy to show that $L_3^{\zeta}(\mu)$ and then $\bar{\partial}\zeta_z(\omega)$, transform like a true $(2, -1)$ differential. Thus, Eq. (3.3) is well defined on the Riemann surface Σ (even if the atlas used is not projective). The same can be verified for Eq. (3.2). Then, for a given Beltrami differential μ, i.e. a given complex structure, the Schwarzian derivative serves to distinguish between projective structures.

4. μ-Holomorphy conditions for *j*-differentials

Here, we consider the operator

$$W_j \equiv \bar{\partial} - \mu\partial - j\partial\mu. \tag{4.1}$$

A solution f_j of the following μ-holomorphy condition:

$$W_j f_j = 0 \tag{4.2}$$

is called a μ-holomorphic j-differential [1]. Then, each j-differential f_j on any two-dimensional Riemann surface Σ is related to the operator W_j by the μ-holomorphy equation (4.2).

Moreover, it is easy to verify that, if f_j is a μ-holomorphic j-differential, $f_j^j \equiv (f_j)^j$ is a μ-holomorphic f^2-differential:

$$W_{j^2} f_j^j = 0. \tag{4.3}$$

Indeed, by direct calculation, we get

$$W_{j^2} f_j^j = jf_j^{j-1} W_j f_j. \tag{4.4}$$

Now, let us give some important examples in conformal geometry and 2D-conformal models for such μ-holomorphy conditions.

4.1. 0-differential

For $j = 0$, the μ-holomorphy condition (4.2) is reduced to the Beltrami equation satisfied by a μ-holomorphic 0-differential; a scalar field $F(z, \bar{z})$:

$$W_0 F = 0. \tag{4.5}$$

Geometrically, the field F parameterizes another complex structure corresponding to a Beltrami differential μ which is defined by (4.5), as $(z, \bar{z})$ is a local coordinates system of the reference complex structure corresponding to $\mu = 0$. In string theory, the field F of Eq. (4.5) is interpreted as the Wess–Zumino field [9].

4.2. $-\frac{1}{2}$-differential

$j = -A/2$ defines a μ-holomorphic $-\frac{1}{2}$-differential $\Psi(z, \bar{z})$ which is a spinor function satisfying

$$W_{-1/2}\Psi = 0. \tag{4.6}$$

The solution $\Psi = (\partial Z)^{-1/2}$ of Eq. (4.6), where Z is a μ-holomorphic 0-differential, is the inverse of the square root of the conformal factor $\lambda \equiv \partial Z$ when the metric is parameterized by a local isothermal coordinates system $(Z(z, \bar{z}), \overline{Z(z, \bar{z})})$ defined by Eq. (2.4).

Moreover, one can verify that the $-\frac{1}{2}$-differential $\Psi = (\partial Z)^{-1/2}$ satisfies the following equation:

$$(\partial^2 + \tfrac{1}{2}\zeta_z(Z))((\partial Z)^{-1/2}) = 0. \tag{4.7}$$

This determines the Schwarzian derivative of the diffeomorphism $z \to Z(z, \bar{z})$.

On the other hand, it is easy to see that Eqs. (4.6) and (4.7) for the μ-holomorphic $-\frac{1}{2}$-differential $\Psi = (\partial Z)^{-1/2}$ are compatible with the μ-holomorphy equation (3.3).

To this end, let us rewrite Eqs. (4.6) and (4.7), respectively, as

$$(\bar{\partial} - h)\begin{pmatrix} \Psi \\ \partial\Psi \end{pmatrix} = 0, \quad h_{11} = -h_{22} = -\tfrac{1}{2}, \quad h_{12} = \mu \text{ and } h_{21} = -\tfrac{1}{2}\mu\zeta_z\partial^2\mu \tag{4.8}$$

and

$$(\bar{\partial} - l)\begin{pmatrix} \Psi \\ \partial\Psi \end{pmatrix} = 0, \quad l_{11} = l_{22} = 0, \qquad l_{12} = 1 \text{ and } l_{21} = -\tfrac{1}{2}\zeta_z. \tag{4.9}$$

Hence, compatibility of Eqs. (4.8) and (4.9) is equivalent to the vanishing of the curvature

$$R_{hl} = [\bar{\partial} - h, \bar{\partial} - l], \tag{4.10}$$

which implies Eq. (3.3).

4.3. 1-differential

The μ-holomorphic 1-differential is a vector field V defined by the μ-holomorphy condition

$$W_1 V = 0. \tag{4.11}$$

The well-known example is the conformal factor $\lambda = \partial Z$. Indeed, this later satisfies

$$W_1\lambda = 0, \tag{4.12}$$

when the Beltrami equation for Z holds.

Moreover, more generally, it is easy to verify that λ^j is a μ-holomorphic j-differential for any real number j, i.e.

$$W_j\lambda^j = 0. \tag{4.13}$$

This latter is deduced from $W_j\lambda^j = j\lambda^{j-1}W_1\lambda$. Hence, Eq. (4.6) for $\Psi = \lambda^{-1/2}$ and (4.12) for λ are particular cases of (4.13), respectively, for $j = -\frac{1}{2}$ and $j = 1$.

4.4. 2-differential

For $j = 2$, an example of a 2-differential is the classical energy–momentum tensor $\Theta_{zz}(z, \bar{z}) \equiv \delta S_C(\mu)/\delta\mu$, where S_C is the classical action of the model under consideration. Indeed, this tensor satisfies the μ-holomorphy condition: $W_2\Theta = 0$ which is the well-known classical Ward identity of a two-dimensional conformal model. In other words, a μ-holomorphy condition for a 2-differential is realized, on the functional space of 2D conformal field theory, as the kernel of the Ward operator W_2. This latter equation expresses the diffeomorphisms invariance of the classical action of the model.

5. Some deformed μ-holomorphy conditions

The first example of such equations is the compatibility condition (3.3) for the Schwarzian derivative $\zeta_z(\omega)$ of a quasi-conformal transformation. This condition can be rewritten as

$$W_2\zeta_z = \partial^3\mu \tag{5.1}$$

which is compatible with the fact that ζ_z is not a 2-differential with respect to a conformal change of coordinates (see Eq. (3.10)).

Another important example is the anomalous conformal Ward identity

$$W_2 T_{zz}(z,\bar{z}) = -\tfrac{1}{2}k\partial^3\mu, \tag{5.2}$$

where T_{zz} is the energy–momentum "tensor" of an effective two-dimensional conformal model: $T_{zz}(z,\bar{z}) \equiv (\delta Z_v^c(\mu)/\delta\mu)\cdot Z_v^c(\mu)$ is the generating connected Green functions of the vacuum.

Here also, Eq. (5.2) is compatible with the fact that the field T_{zz} is not conformally covariant. Indeed, under a holomorphic change of coordinates T_{zz} transforms as

$$z \to \omega(z), \qquad T_{zz} \to T_{\omega\omega} = (\partial\omega)^{-2}(T_{zz} + \tfrac{1}{2}\zeta_z(\omega)) \tag{5.3}$$

Another type of deformed μ-holomorphy condition is given by the following:

$$W_{-1}C = s\mu, \tag{5.4}$$

which is the BRST transformation of the Beltrami differential μ. However, when the conformal gauge [9]

$$s\mu = 0 \tag{5.5}$$

is considered, the ghost field C becomes a μ-holomorphic 1-differential. Knowing that the gauge fixing action of the bosonic string is given by

$$S_{\mathrm{GF}} = \int_\Sigma \mathrm{d}m\, bs\mu + \mathrm{c.c.}, \tag{5.6}$$

where $\mathrm{d}m(z) \equiv \mathrm{d}\bar{z}\wedge\mathrm{d}z/2\mathrm{i}$ is the measure on the Riemann surface Σ and b is an auxiliary field, the conformal gauge (5.5) is equivalent to the equation of motion of the field b which is a non dynamical equation and enables to eliminate the field b from the action.

On the other hand, any projective connection R on Σ satisfies the μ-holomorphy equation [14]

$$\bar{\partial}R = L_3^R(\mu), \tag{5.7}$$

which can rewritten as a certain deformed μ-holomorphy condition for the projective connection

$$W_2 R = \partial^3 \mu. \tag{5.8}$$

This equation is also compatible with the fact that R is not a tensor under a holomorphic change of coordinates

$$R_{\omega\omega} = (\partial\omega)^{-2}(R_{zz} - \zeta_z(\omega)) \tag{5.9}$$

to be compared with Eq. (3.6). This implies that the general solution of the μ-holomorphy equation (5.8) is given by

$$R_{zz}(z, \bar{z}) = \zeta_z(Z) + f_{zz}(z, \bar{z}), \tag{5.10}$$

where f is a 2-differential. Moreover, f can be expressed as

$$f(z, \bar{z}) = \lambda^2 Q(Z), \tag{5.11}$$

where Q is holomorphic in the complex structure $(Z, \bar{Z}) : \partial_{\bar{Z}} Q = 0$. At the end, we conclude that the μ-holomorphy condition, in the reference structure, is equivalent to the holomorphic condition in the complex structure $(Z, \bar{Z})$.

6. Conclusion and open problems

The construction of j-differentials on a Riemann surface Σ enables to get the conformally covariant objects which are needed to get the model globally defined on Σ. On the other hand, the transformation

$$\begin{aligned} R_0 \to R, \qquad \bar{\partial} R_0 = 0 \to \bar{\partial} R = L_3^R(\mu), \qquad sR_0 = 0 \to sR = L_3^R(C), \\ s\mu_0 = 0 \to s\mu = (\bar{\partial} - \mu\partial + \partial\mu)C \end{aligned} \tag{6.1}$$

shed some light on the geometric representation of the BRST operator. Here, in Eq. (6.1), the BRST operator s plays the same role as the co-bord operator $\bar{\partial}$.

References

[1] N. Seiberg, in: Proceedings of the Lecture at the Yukawa International Seminar on Common Trends in Mathematics and Quantum Field Theories, Kyoto, Japan, 1990 and at the Cargese Meeting Random Surfaces, Quantum Gravity and Strings, Cargese, France, 1990 and references there-in.

[2] H. Verlinde, Conformal field theory, two-dimensional quantum gravity and quantization of Teichmuller space, Nucl. Phys. B 337 (1990) 652.

[3] A.M. Polyakov, Quantum geometry of the bosonic strings, Phys. Lett. B 103 (1981) 207.

[4] D. Friedan, in: R. Stora, J.B. Zuber (Eds.), Recent Advances in Field Theory and Statistical Mechanics, Proceedings of the Les Houches Summer School, North-Holland, Amsterdam, August/September 1982.

[5] M. Knecht, S. Lazzarini, F. Thuillier, Shifting the Weyl anomaly to the chirally split diffeomorphism anomaly in two dimensions, Phys. Lett. B 251 (1990) 279.

[6] A.A. Belavin, K.G. Kniznik, Algebraic geometry and the geometry of quantum strings, Phys. Lett. B 168 (1986) 20.
[7] M. Knecht, S. Lazzarini, R. Stora, On holomorphic factorization for free conformal field II, Phys. Lett. B 273 (1991) 63.
[8] A.M. Polyakov, Quantum gravity in two dimensions, Mod. Phys. Lett. A 2 (1987) 893.
[9] S. Lazzarini, On bi-dimensional Lagrangian conformal models, Doctoral Thesis, LAPP, Annecy-LeVieux, 1990.
[10] F. Gieres, Conformaly covariant operators on Riemann surfaces (with application to conformal and integrable models), Int. J. Mod. Phys. A 8 (1) (1993) 1.
[11] M. Kachkachi, M. Kessabi, Conformal Ward identity and μ-holomorphic equation relationship, RNPT/1/LPHER/13/2000, J. Math. Phys., submitted for publication.
[12] L. Bonora, P. Cotta-Ramusino, C. Riena, Conformal anomaly and cohomology, Phys. Lett. B 126 (1983) 305.
[13] L. Baulieu, M. Bellon, Beltrami parameterization in string theory, Phys. Lett. B 196 (1987) 142.
[14] R. Zuccsini, A Polyakov action on Riemann surfaces (II), Commun. Math. Phys. 152 (1993) 269.